Carleton UNIVERSITY
Canada's Capital University

Mathematics Tutorial Series
Differential Calculus

Trigonometric Functions II

Trig derivatives – sin and cos

1.
$$\frac{d \sin x}{dx} = \cos x$$
2.
$$\frac{d \cos x}{dx} = -\sin x$$

Reasons:

1. We have to prove at least one trig derivative from scratch. Usually, this is $sin\ x$.

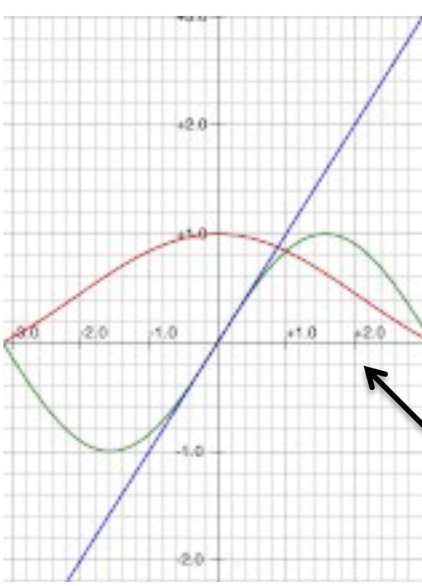
Remember the definition: The value of the derivative of f at a is:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

When $f(x) = \sin x$ we set $h = x - a$ and $x = a + h$. This sets us up for the trig identity for sin of an angle sum. At the end of these notes there is an outline of the proof for the derivative of $\sin x$. At the heart of the proof is a classic limit:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

This limit is a fact about Euclidean geometry with many consequences, not least the derivative of $sin\ x$. We must use radians to make this limit true.



Red is $y = \frac{\sin h}{h}$

Blue is $y = h$

Green is $y = sin\ h$

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

h-axis

2. Derivative of *cos x*.

We can write the identity:

$$\cos^2 x + \sin^2 x = 1$$

as

$$(cos\, x)^2 = 1 - (\sin x)^2.$$

Then apply the chain rule:

$$2(\cos x)\frac{d\cos x}{dx} = 0 - 2(\sin x)\frac{d\sin x}{dx}$$

We already know that

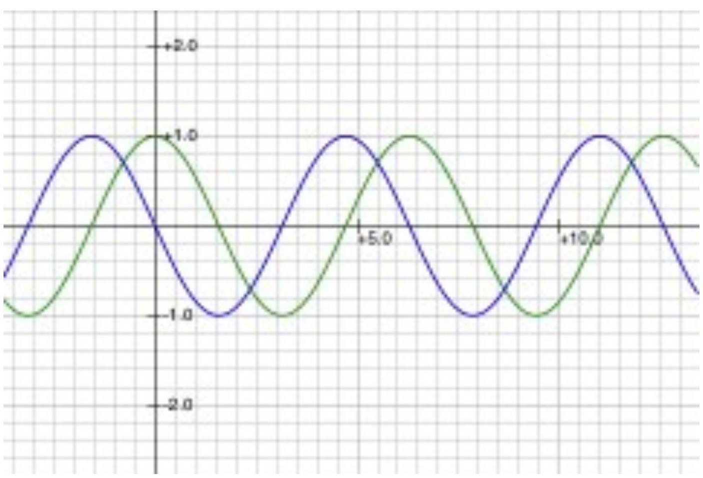
$$\frac{d\sin x}{dx} = \cos x$$

So we have

$$2(\cos x)\frac{d\cos x}{dx} = 0 - 2(\sin x)\,(\cos x)$$

Which simplifies to

$$\frac{d\,\mathbf{cos\,x}}{dx} = -\,\mathbf{sin}\,\boldsymbol{x}$$



Summary

1. Derivatives: $\frac{d \sin x}{dx} = \cos x$ and $\frac{d\, cos\, x}{dx} = -\sin x$
2. In calculus, always work in radians.
3. $\lim_{h \to 0} \frac{\sin h}{h} = 1$

Outline of the Proof that $\frac{d \sin x}{dx} = \cos x$

We have already

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

We also need

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

To see this:

$$\frac{\cos h - 1}{h}$$

$$= \frac{(\cos h - 1)(\cos h + 1)}{h\,(\cos h + 1)}$$

$$= \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \frac{-\sin^2 h}{h(\cos h + 1)}$$

$$= \frac{1}{\cos h + 1} \cdot \frac{\sin h}{h} \cdot (-\sin h) \to 0$$

Then use $\sin(a + h) = \sin a \cos h + \cos a \sin h$ to get

$$\frac{\sin(a + h) - \sin a}{h}$$

$$= \frac{\sin a \cos h + \cos a \sin h - \sin a}{h}$$

$$= \sin a\, \frac{\cos h - 1}{h} + \cos a \frac{sin\ h}{h}$$

Now we let $h \to 0$ and the derivative of $sin\ x$ at a is $cos\ a$.